QUANTIZATION OF SOLITONS

L. D. Faddeev

Steklov Mathematical Institute, Leningrad, USSR

and Institute for Advanced Study, Princeton, New Jersey 08540, USA

This paper contains the main results and techniques presented in my lectures at Harvard and Princeton in May 1975. These lectures were based on work by me and my collaborators I. Arefieva, L. A. Takhtajan, V. E. Korepin and P. P. Kulish published in (1) - (7) and had either a long title, "Localized Solutions of the Classical Field Equations and their Quantum Interpretation" or a short one, "Quantization of Solitons," which contain the same information. The problem treated in these lectures has recently attracted the attention of several groups and the number of papers or preprints is increasing rapidly. The list (8) - (20) is a more or less accurate description up to April 1975. I shall not make an extensive comparison of the existing approaches with the one presented here but confine myself to several comments in suitable places. Some errors in (4) are corrected.

The technique will be illustrated on the example of the sine-Gordon equation. It will be clear from the text which results have general validity and which use the particular structure of this beautiful field theoretical model.

The starting point of the discussion is a Hamiltonian formulation of the corresponding classical problem. The inverse scattering method (see i. e. (21)) applied to the sine-Gordon equation (2), (3), allows one to describe and parametrize all classical solutions with finite energy. In particular, expressions for observables such as energy and momentum in terms of angle action variables (2) display the particle spectrum of the system (§ 1) and allow one to calculate the S-matrix for solitons (§ 3) in a quasiclassical approximation. A systematic procedure for computing the quantum corrections to the quasiclassical results will be developed in § 2. Some comments about the way to generalize the notion of a soliton to higher dimensions and a conclusion will be presented in § 4.

During my stay in the USA I have benefitted from discussions with C. Callan, S. Coleman, R. Dashen, S. Drell, D. Gross, B. Hasslacher, R. Jackiw, A. Neveu and I am happy to have an opportunity to thank them. B. Hasslacher's help with the English translation cannot be overestimated.

1. THE DESCRIPTION OF THE CLASSICAL SYSTEM

Let $\chi(x,t)$ be a chiral field in two dimensional space-time,

$$\chi(x,t) = \exp\{iu(x,t)\}$$

satisfying the boundary condition

$$\chi(x,t) \to 1, \qquad |x| \to \infty$$

so that

$$u(x,t) \to 0 \pmod{2\pi}, \qquad |x| \to \infty$$

The integer

$$Q = \frac{1}{2\pi}\left(u(\infty,t) - u(-\infty,t)\right)$$

is a charge associated with the current

$$J_\mu = \frac{1}{2\pi}\, \varepsilon_{\mu\nu}\, \partial_\nu u - \frac{1}{2\pi}\, \varepsilon_{\mu\nu}\, \partial_\nu \chi\, \chi^{-1},$$

which is conserved independently of the equations of motion.

The Lagrangian of the sine-Gordon equation has the form

$$\mathcal{L} = \frac{1}{\gamma} \int_{-\infty}^{\infty} \left[\frac{1}{2}\left(u_t^2 - u_x^2\right) - m^2\left(1 - \cos u\right)\right] dx$$

Here m is the mass of the field u, γ is a dimensionless coupling constant. We use the usual convention $h = 1$, $c = 1$.

The classical equations of motion and the Poisson brackets look as follows

$$u_{tt} - u_{xx} + \sin u = 0,$$

$$\{\pi(x,t),\ u(y,t)\} = \delta(x-y); \qquad \pi(x,t) = \frac{1}{\gamma}\, u_t(x,t)$$

The inverse scattering method applied to these equations defines a canonical transformation of the phase-space of the variables $u(x)$, $\pi(x)$ onto a phase space parametrized by the scattering data for the auxiliary linear problem which has the following structure:

1. Continuous spectrum, $\rho(p)$, $\varphi(p)$, $-\infty < p < \infty$

$$0 \le \rho(p) < \infty; \quad 0 \le \varphi(p) \le 2\pi; \quad \{\varphi(p), \rho(p')\} = \delta(p-p')$$

2. Solitons, p_a, q_a, $a = 1, \ldots, A$, A is an arbitrary integer

$$-\infty < p_a < \infty; \ -\infty < q_a < \infty$$

$$\{p_a, q_{a'}\} = \delta_{aa'}.$$

3. Double solitons (breathers)

p_b, q_b, α_b, β_b, $b = 1, \ldots, B$, B is an arbitrary integer

$$-\infty < p_b < \infty; \qquad -\infty < q_b < \infty,$$

$$0 \le \alpha_b \le \pi/2; \qquad 0 \le \beta_b \le \frac{32\pi}{\gamma};$$

$$\{p_b, q_{b'}\} = \delta_{bb'}; \qquad \{\alpha_b, \beta_{b'}\} = \delta_{bb'}$$

We have written explicitely all the non-trivial Poisson brackets.

The energy and momentum of the field is given by the expressions

$$H = \frac{1}{\gamma}\int_{-\infty}^{\infty}\left[\frac{1}{2}\left(u_t^2 + u_x^2\right) + m^2\left(1-\cos u\right)\right] dx =$$

$$= \int_{-\infty}^{\infty} \sqrt{p^2+m^2}\,\rho(p)\,dp + \sum_a^A \sqrt{p_a^2 + M^2} + \sum_b^B \sqrt{p_b^2 + M^2(\alpha_b)};$$

$$P = \frac{1}{\gamma}\int_{-\infty}^{\infty} u_t u_x \, dx =$$

$$= \int_{-\infty}^{\infty} p\,\rho(p)\,dp + \sum_a^A p_a + \sum_b^B p_b,$$

where

$$M = \frac{8m}{\gamma}; \qquad M(\alpha) = 2M \sin\alpha = \frac{16m}{\gamma}\sin\alpha,$$

and it is clear why we call the scattering data parametrization an "angle-action" form of phase space. Indeed, energy and momentum are functions of the generalized momenta in this representation.

Quantization in the quasiclassical-tree approximation (to distinguish it from true quasiclassics which includes one loop) can be achieved by changing Poisson brackets into corresponding commutators. The operator $\rho(p)$ is conjugate to the "angle" operator so that its eigenvalues assume the form

$$\rho'(p) = \sum_{c}^{C} \delta(p - p_c)$$

the integer C and the numbers p_c being arbitrary. The contribution of the continuous spectrum to the energy is that of a scalar particle of mass m--namely the fundamental particle of the theory. Its charge Q is zero.

The variables p_a, q_a for solitons quantize trivially. The spectrum of p_a is the whole real line and so its contribution to the energy is that of a particle of mass M. Its charge, $Q = \pm 1$ which is obvious from the form of the classical solution in the case $\rho = 0$, $B = 0$, $A = 1$

$$u_1(x,t \mid p, q, \pm) = 4 \operatorname{arctg}\left(\exp\left\{ \pm m \frac{x - v t - q}{\sqrt{1 - v^2}} \right\}\right), \qquad v = \frac{p}{\sqrt{p^2 + M^2}} .$$

The double soliton has two degrees of freedom: p, q correspond to the center of mass motion and α, β to internal motion. The mass of a double soliton

$$M(\alpha) = 2M \sin\alpha$$

depends on this motion and can assume any value from 0 to 2M. The

classical solution for the case $\rho = 0$, $A = 0$, $B = 1$

$$u_2(x, t \mid p, q, \alpha, \beta) =$$

$$= 4 \operatorname{arctg}\left\{ \operatorname{tg}\alpha \, \frac{\cos\left[m \cos\alpha \, (x \operatorname{sh}\psi - t \operatorname{ch}\psi - \beta)\right]}{\operatorname{ch}\left(m \sin\alpha \left[(x - q) \operatorname{ch}\psi - t \operatorname{sh}\psi \right]\right)} \right\}$$

where ψ is defined through

$$p = 2M \sin\alpha \operatorname{sh}\psi$$

makes it obvious that the charge Q for a double soliton is 0.

When quantizing one must pay attention to the peculiar character of the phase space for the internal motion, namely that it is compact. The full volume of this phase space is equal to $16\pi^2/\gamma$ and this is to be set equal to $2\pi N$ where N is the number of states in the quantized version. The condition

$$\frac{16\pi^2}{\gamma} = 2\pi N , \qquad N = \frac{8\pi}{\gamma}$$

is valid approximately for large N or small γ. The possible eigenvalues of α are to be quantized also and in the first approximation are equal to

$$\alpha_n = \frac{\pi}{2N} n , \qquad n = 1, \dots, N$$

(at least for n large enough) so that we have a mass formula

$$M_n = \frac{16 m}{\gamma} \sin \frac{n\gamma}{16}$$

for the masses of the double soliton excitations in quantum mechanics. The formal limit $\gamma \to 0$, $n = 1$ gives the value

$$M_1 \equiv m$$

for the mass of the lowest excitation, coinciding with the mass of the fundamental particles. R. Dashen, B. Hasslacher and A. Neveu (DHN in what follows) have conjectured that this mass formula is exact after a renormalization of γ, coming from the one-loop correction. This observation makes one believe that the full quantum mechanical spectrum is generated by soliton and double soliton degrees of freedom. The following experiment shows that this surprizing result can happen to be true.

Let us consider a non-relativistic example given by the nonlinear Schroedinger equation

$$i\psi_t = -\psi_{xx} + 2|\psi|^2\psi$$

which is a classical equation for a second quantized system of particles with a pairwise interaction via a δ-function potential. This equation is completely integrable (22) in classical form and exactly soluble in quantum mechanics (see i.e. (23)). There is only one type of soliton solution and it has two degrees of freedom. The corresponding phase space coordinates can be taken in the form

$$p_a,\ q_a,\ \eta_a,\ \varphi_a\ , \quad -\infty < p_a < \infty\ ; \quad -\infty < q_a < \infty\ ;$$

$$0 \le \eta_a < \infty\ ; \qquad 0 \le \varphi_a \le 2\pi\ ,$$

$a = 1, \dots, A$, A is an arbitrary integer.

There exists also a continuous spectrum, characterized by functions $\rho(p)$, $\varphi(p)$, analogous to the case of the sine-Gordon equation. The contribution to the observables such as the number of particles N, momentum P, and energy H assumes the form (see [22])

$$N = \int_{-\infty}^{\infty} \psi^* \psi \, dx = \int_{-\infty}^{\infty} \rho(p)\, dp + \sum_a^A \eta_a\ ;$$

$$P = \frac{1}{2i}\int_{-\infty}^{\infty} (\psi^*_x \psi - \psi^* \psi_x)\, dx = \int_{-\infty}^{\infty} p\rho(p)\, dp + \sum_a^A p_a\ ;$$

$$H = \int_{-\infty}^{\infty} (\psi^*_x \psi_x - |\psi|^4)\, dx = \int_{-\infty}^{\infty} p^2\rho(p)\, dp + \sum_a^A \left(\frac{p_a^2}{\eta_a} - \frac{\eta_a^3}{48}\right)$$

In quantum mechanics one uses an expression for H in normal ordered form which can differ from H for a different choice of ordering of factors by the addition of some multiple of N

$$H_{norm} = H + cN\ ,$$

c depends on the choice of ordering in H.

After quantizing, the eigenvalues of η assume the form (in tree approximation)

$$\eta'_k = k\ , \quad k = 1, \dots$$

which is valid in general for k large enough. We shall see however that this answer is exact after fixing the constant c. Indeed the contribution of the solitons to the energy and momentum looks as follows

$$P = \sum_a^A p_a \ ; \qquad H_{norm} = \sum_a^A \left(\frac{p_a^2}{k_a} - \mathcal{E}_{k_a} \right) , \quad \mathcal{E}_k = \frac{k^3}{48} - ck \ .$$

We see that these contributions are those of particles of mass $k/2$ and a dispersion law for the corresponding energy

$$\omega_k(p) = \frac{p^2}{k} - \mathcal{E}_k$$

The fundamental particle of mass $1/2$ appears for $k = 1$. To have a usual dispersion law $\varepsilon(p) = p^2$ for it one must choose $c = 1/48$. With this choice, the energy of the excitation with $k > 1$ becomes

$$\omega_k(p) = \frac{p^2}{k} - \Delta_k \ ; \qquad \Delta_k = \frac{1}{48}(k^3 - k)$$

and can be interpreted as the energy of a bound state of k particles with binding energy Δ_k. It is known that these energies exhaust the full spectrum of H_{norm} and so it is clear that one must not include the continuous part of the classical phase space to obtain the spectrum of the quantum Hamiltonian. The reason for this phenomenon is still unclear, but at least it confirms the DHN conjecture that it is enough to count only the soliton degrees of freedom to find the full quantum mechanical spectrum of the sine-Gordon equation.

We return now to the main subject with the comment that the classical sine-Gordon equation has an infinite number of local conservation laws. We shall not give the explicit form of the corresponding densities, since recurrence relations defining them can be found in (2). It will be

enough to note that they are the deformations of analogous laws for the free Klein-Gordon equation

$$u_{tt} - u_{xx} + m^2 u = 0 \ ,$$

which have the form

$$P_{2n}^{(0)} = \frac{1}{2} \int_{-\infty}^{\infty} \left[\left(\left(\frac{\partial}{\partial x}\right)^n u_t \right)^2 + \left(\left(\frac{\partial}{\partial x}\right)^n u_x \right)^2 + m^2 \left(\left(\frac{\partial}{\partial x}\right)^n u \right)^2 \right] dx \ ;$$

$$P_{2n+1}^{(0)} = \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial x}\right)^n u_t \left(\frac{\partial}{\partial x}\right)^n u_x \, dx \ , \qquad n = 0, 1, 2, \ldots$$

The conservation laws P_{2n}, P_{2n+1} for the sine-Gordon equation differ from these by the addition of local densities of order 3 and higher in u and its derivatives. The mere existence of such conservation laws causes a severe restriction on the scattering. The variety of arguments below belong to A. Polyakov.

Let us express the conservation laws in terms of in or out variables. In the limit $t \to \pm\infty$ the nonquadratic terms in P_{2n-1}, P_{2n} vanish so that

$$P_k(\pi, u) \to P_k^{(0)}(\pi_{\substack{out\\in}}, u_{\substack{out\\in}}) \ ,$$

and the asymptotic expressions on the RHS are equal. This produces a series of identities of the form

$$\sum_a (p_{0a} p_a^{2n})_{in} = \sum_a (p_{0a} p_a^{2n})_{out} \ ;$$

$$\sum_a (p_a^{2n+1})_{in} = \sum_a (p_a^{2n+1})_{out} \ .$$

where the sum is taken over all types of initial and final particles. The consequence is that the number of the particles in each process, as well as their types and momenta are conserved. This means that the S-matrix is proportional to the unit operator

$$S = I \exp\{-2i\delta\}$$

where

$$I = \mathrm{sym} \prod_{a} \delta(p_{in} - p_{out})$$

with symmetrization for each type of particle. The net effect of the scattering consists of a phase shift δ, which depends on the momenta of particles in the process.

2. PERTURBATION THEORY

Let us now consider the quantum corrections to the classical results described above. It is very convenient to use a functional integral method to compute them. Conservation of the charge Q is also evident in the functional integral formulation. Indeed, fields with different Q are nonhomotopic, that is they cannot be deformed one into another in a continuous way.[24] This means that the action for any trajectory connecting such fields is infinite and the transition amplitude vanishes.

We shall illustrate the perturbation theory by computing the correction to the soliton mass. We shall use a one-particle Green's function

$$G(p_2,t_2|p_1,t_1) = \int \psi^*_{p_2}(u) \exp\left\{ i \int_{t_1}^{t_2} (\pi u_t - \mathcal{H}(u,\pi))\, dx\, dt \right\} \psi_{p_1}(u) \prod_{x,t} du\, d\pi$$

to do this. Here $\mathcal{H}(u,\pi)$ is the energy density and $\psi_p(u)$ is a wave functional for a one soliton state with momentum p in a Schroedinger-like representation. The Green's function must have the form

$$G(p_2,t_2|p_1,t_1) = F\, \delta(p_2 - p_1)\ ; \quad F = \exp\left\{-i\sqrt{p^2 + M_q^2}\; T\right\} \quad (*)$$

$$T = t_2 - t_1$$

where M_q is the exact quantum mass of the soliton. The expression for G above is useless as it stands, because we do not know the explicit form of the wave functional. This difficulty can be circumvented however if we consider the limit of $G(p_2, t_2|p_1, t_1)$ for $t_1 \to -\infty$, $t_2 \to \infty$, so that $T \to \infty$. An example from the quantum mechanics of a particle with one degree of freedom shows that $G(p_2,t_2|p_1,t_1)$ can be found in this asymptotic region, if we begin with the Green's function $G(x_2, t_2|x_1, t_1)$ in the coordinate representation and put $x_1 = \frac{p_1}{m} t_1 + q_1$, $x_2 = \frac{p_2}{m} t_2 + q_2$. For large $|t_1|$ and $|t_2|$ the expression we get in this way does not depend on q_1 and q_2 and is proportional to $G(p_2,t_2|p_1,t_1)$.

The analogous recipe in the one soliton case looks as follows. The Green function $G(p_2 t_2|p_1 t_1)$ for $t_1 \to -\infty$, $t_2 \to \infty$ is given by the functional integral

$$\int \exp\left\{ i \int_{t_1}^{t_2} [\pi u_t - \mathcal{H}(u,\pi)]\, dx\, dt \right\} \prod_{x,t} du\, d\pi \,,$$

where we integrate over all fields $u(x,t)$ and $\pi(x,t)$ such that

$$u(x,t)\big|_{t=t_1} = u_1(x,t_1|p_1,q_1)\ ; \quad u(x,t)\big|_{t=t_2} = u_2(x,t_2|p_2,q_2)\,,$$

where $u_1(x,t|\,p,q)$ is a one soliton classical solution. In the limit $t_1 \to -\infty$, $t_2 \to \infty$ this expression ceases to depend on q_1 and q_2 and must assume the form (*).

It is easy to understand why this limit is proportional to a δ-function. Indeed, if $p_1 \neq p_2$ there exists no classical trajectory with the prescribed values for $t = t_1$ and $t = t_2$, so that the functional integral vanishes (at least in the stationary phase approximation). For $p_1 = p_2 = p$ there exists an infinite number of admissible trajectories, any one soliton solution $u_1(x,t|p,q)$ can be used as such. The corresponding action does not depend on q and the functional integral becomes proportional to

$$\int_{-\infty}^{\infty} dq = 2\pi\,\delta(0)$$

which concludes the argument.

It is clear that one must modify this method of computing the functional integral and try to obtain directly the coefficient F in front of the δ-function in (*). The degeneracy mentioned above can be remedied if we can integrate over fields with fixed total momentum p to obtain the transition amplitudes in the invariant subspace defined by

$$P(u,\pi) = \int \pi u_x \, dx = p$$

The general recipe for performing functional integrals on systems with constraints (25) shows that F is given by the expression

$$\int \exp\left\{ i \int_{t_1}^{t_2} [\pi u_t - \mathcal{H}(u,\pi)]\,dx\,dt \right\} \prod_t \delta\big(P(u,\pi)-p\big)\,\delta\big(X(u,\pi)\big)\,\{P,X\} \prod_{x,t} du\,d\pi$$

where

$$u(x,t)\big|_{t=t_1} = u_1(x,t_1|\,p,q_1)\ ; \qquad u(x,t)\big|_{t=t_2} = u_1(x,t_2|\,p,q_2)$$

in the limit $t_2 \to \infty$, $t_1 \to -\infty$, when it does not depend on q_1 and q_2, so that to compute, one can put $q_1 = q_2 = 0$. Here $X(u,\pi)$ is an arbitrary subsidary condition such that

$$\{ P(u,\pi),\ X(u,\pi) \} \neq 0 .$$

The integral does not depend on the choice of X, and is covariant, so that

$$F(p,T) = F\left(\sqrt{1-v^2}\,T\right), \qquad v = \frac{p}{\sqrt{p^2+M^2}}$$

We shall proceed to compute F for $v = 0$ which simplifies the procedure.

The subsidary condition

$$X(u,\pi) = \frac{\int_{-\infty}^{\infty} x\,\mathcal{H}(u,\pi)\,dx}{\int_{-\infty}^{\infty} \mathcal{H}(u,\pi)\,dx} - f(t)$$

where $f(t) \to 0$, $|t| \to \infty$ is convenient because $\{P, X\} = 1$ for such an X and we shall use it below. We make use of the stationary method, beginning with the change of variables

$$u(x,t) = u(x) + \sqrt{\gamma}\, z(x,t)\ ; \qquad \pi = \frac{1}{\sqrt{\gamma}}\, w(x,t)$$

where

$$u(x) = u_1(x,t|\,0,0)$$

is independent of t. The functions $z(x,t)$, $w(x,t)$ will play the role of the deviation from a given classical field $u_1(x)$, which is the only admissible stationary trajectory. Confining ourselves to the first two orders in γ we have

$$F(T) \cong F_{-1}(T) \cdot F_0(T)$$

where

$$F_{-1}(T) = \exp\{-iMT\} = \exp\{-i\,\frac{8m}{\gamma}\,T\}$$

and

$$F_0(T) = \int \exp\left\{ i \int_{t_1}^{t_2} \left(w z_t - \frac{1}{2} w^2 - \frac{1}{2} z K z\right) dx\, dt \right\} \times$$

$$\prod_t \delta\left(\int_{-\infty}^{\infty} w u_x\, dx\right) \delta\left(\int_{-\infty}^{\infty} x \left(z_x u_x + v''(u) z\right) dx\right) \prod_{x,t} dz\, dw ,$$

where we use the notation

$$v(u) = m^2 (1 - \cos u)$$

to make it clear that our considerations are general and K is a Schroedinger operator

$$K = -\frac{d^2}{dx^2} + v''(u)$$

The argument of the second δ-function can be transformed in the following way

$$\int_{-\infty}^{\infty} x (z_x u_x + v''(u) z) dx = -\int_{-\infty}^{\infty} z u_x\, dx + \int_{-\infty}^{\infty} x(-u_{xx} + v''(u))\, z\, dx$$

and the second term on the RHS vanishes because of the classical equation of motion for u . We see that the second δ-function now looks analogous to the first one. It is a general fact that u_x is an eigenfunction of K with zero eigenvalue (see i.e. (10)) which is also the only one. The Gaussian integral for F_0 can now be written now in the form

$$F_0(t) = \exp\left(-\frac{1}{2} \mathrm{Tr}' \ln\left(\frac{d^2}{dt^2} + K\right)\right)$$

where the trace is taken in x t space, $-\infty < x < \infty$, $t_1 < t < t_2$ and Tr' indicates that the singular contribution of the zero eigenvalue of K must be dropped.

The formal evaluation of F_0 can be done in the following way. Consider

$$L_\varepsilon = \mathrm{Tr}' \ln \left(\frac{d^2}{dt^2} + \varepsilon K\right)$$

Differentiating

$$\frac{dL_\varepsilon}{d\varepsilon} = \mathrm{Tr}'\left[\left(\frac{d^2}{dt^2} + \varepsilon K\right)^{-1} K\right] =$$

$$= \int_{t_1}^{t_2} dt\ \mathrm{tr}'\left(\frac{e^{i\sqrt{\varepsilon K}(t-t')}}{2i\sqrt{\varepsilon K}} \Big|_{t=t'} K \right)$$

where "tr" is a trace over x-space. After observing that K does not depend on t we obtain

$$\frac{dL_\varepsilon}{d\varepsilon} = -\,i\,T\ \mathrm{tr}'\sqrt{K}\ \frac{1}{2\sqrt{\varepsilon}}$$

and after integrating back, find

$$L_1 = T\mathrm{r}_2'\ \ln\left(\frac{d^2}{dt^2} + K\right) = -i\left(\mathrm{tr}'\sqrt{K} - \mathrm{tr}\sqrt{K_0}\right)T$$

where $K_0 = -\frac{d^2}{dx^2}$ and we have fixed the integration constant so as to subtract the vacuum energy.

To compute the trace one can use the trace formula (see i.e. (26))

$$\mathrm{tr}\left(W(K) - W(K_0)\right) = \frac{1}{2\pi i}\int W'(\lambda)\ln\det S(\lambda) + \sum_n W'(\lambda_n)$$

valid for an arbitrary function $W(\lambda)$. Here $S(\lambda)$ is the S-matrix of K and the sum is over discrete eigenvalues. In our example

$$V(u) = m^2(1-\cos u)\,,\qquad V''(u) = m^2 - \frac{2m^2}{(\mathrm{ch}\,mx)^2}$$

and K has the only one discrete eigenvalue $\lambda = 0$. The S-matrix is given by

$$S = \begin{pmatrix} S_\lambda & 0 \\ 0 & S_\lambda \end{pmatrix}\,,\qquad S_\lambda = \frac{\sqrt{\lambda - m^2} + im}{\sqrt{\lambda - m^2} - im}$$

(no reflection). Using all these formulas we obtain

$$L_1 = \left(\frac{1}{\pi^2 i}\, m\,\mathcal{D}(0) + \frac{2m}{\pi i}\right)T\,,\qquad \mathcal{D}(0) = \int \frac{d^2k}{k^2+m^2}$$

so that

$$F(T) \cong F_{-1}\cdot F_0 = \exp\left\{-i\left(\frac{8m}{\gamma} - \frac{m\,\mathcal{D}(0)}{2\pi^2} - \frac{m}{\pi}\right)\right\}$$

The infinite term is absorbed in the renormalization of the mass m (see (8), (9), (27)). Indeed, the one loop contribution in the mass renormalization of the fundamental particle in ordinary perturbation theory is given by

$$\delta m = -\gamma\frac{m}{16\pi^2}\mathcal{D}(0)$$

so that

$$\frac{8m}{\gamma} - \frac{m\,\mathcal{D}(0)}{2\pi^2} = \frac{8m_r}{\gamma}\,.$$

We see that

$$M_q = \frac{8m_r}{\gamma} - \frac{m_r}{\pi} + 0(\gamma)\,.$$

The sum of the first two terms vanishes when $\gamma = 8\pi$. This value of γ is known to be a critical one for the sine-Gordon equation, see (13). One can speculate that it is an indication that the formula

$$\frac{M_r}{m_r} = \frac{8}{\gamma} - \frac{1}{\pi}$$

for the quotient of the soliton mass and the mass of the fundamental particle is exact. This conjecture, made by DHN (11) is not completely unexpected. Indeed the classical sine-Gordon equation is a completely integrable Hamiltonian system. Quasiclassical results (tree + one loop) for a discrete spectrum are often exact for such systems. To verify the conjecture, one must procede to the next order in γ. This is difficult, though possible and any available simplification of the formalism must be used. For instance, 't Hooft's trick, namely, inserting the constraint and the subsidiary condition into the exponential of the functional integral may be quite useful.

The main result of this section is the statement that quantum corrections for small γ are given by a series in positive powers of γ. The main nonanalytic part $O(\gamma^{-1})$ is supplied by classical field theory.

3. THE S-MATRIX

The existence of an exact classical description of a many soliton motion enables us to calculate the classical (tree) approximation to the S-matrix for solitons. We shall illustrate this with the example of two soliton scattering. Let us consider first, the case of solitons of opposite charge ($s\bar{s}$ scattering). The classical equation of motion has a solution

$$u(x,t \mid p_1, q_1, p_2, q_2, +, -) =$$

$$= 4 \operatorname{arctg} \left(\operatorname{cth} \frac{\psi_1 - \psi_2}{2} \; \frac{\operatorname{sh} \frac{d_1 - d_2}{2}}{\operatorname{ch} \left(\frac{d_1 + d_2}{2} - \ln \operatorname{th} \frac{\psi_1 - \psi_2}{2} \right)} \right)$$

where

$$d_a = m \operatorname{ch} \psi_a (x - q_a) - m \operatorname{sh} \psi_a t ; \qquad p_a = M \operatorname{sh} \psi_a ,$$

with the following asymptotics for $|t| \to \infty$

$$u_2 \to \begin{cases} u_1(x,t \mid p_1^-, q_1^-, +) + u_1(x,t \mid p_2^-, q_2^-, -) , & t \to -\infty \\ u_1(x,t \mid p_1^+, q_1^+, +) + u_1(x,t \mid p_2^+, q_2^+, -) , & t \to \infty \end{cases}$$

The in and out variables, p^-, q^-, p^+, q^+ are connected by the relations

$$p_1^+ = p_1^- = p_1 ; \qquad p_2^+ = p_2^- = p_2 ;$$

$$q_1^+ = q_1^- + \frac{\partial K}{\partial p_1} ; \qquad q_2^+ = q_2^- + \frac{\partial K}{\partial p_2} ,$$

which can be considered as a canonical transformation. The corresponding generator $K(p_1 p_2)$ is defined by the formula

$$K(p_1, p_2) = K(\xi) = \frac{16}{\gamma} \int_1^{\xi} \frac{dx}{x} \ln \frac{x-1}{x+1} = \frac{8i}{\gamma} \int_0^{\pi} d\theta \ln \frac{\xi + e^{i\theta}}{\xi e^{i\theta} + 1}$$

$$\xi = \frac{s - 2M^2 + \sqrt{s(s-4M^2)}}{2M^2} ; \quad s = (p_{10} + p_{20})^2 - (p_1 + p_2)^2 \geq 4M^2 .$$

The branch appearing in the definition of ξ is to be taken in such a way that $s \to \xi$ maps the physical sheet of the s-plane with the cuts $-\infty < s < 0$; $4M^2 < s < \infty$ onto the upper half plane. The integration constant in the definition of K is chosen to make $K(1) = 0$.

The quantum unitary operator, corresponding to this canonical transformation is a classical-tree approximation to the S-matrix

$$\langle p_1', p_2' | S_{s\bar{s}} | p_1, p_2 \rangle = \delta(p_1 - p_1')\,\delta(p_2 - p_2')\, S_{s\bar{s}}(s)$$

$$S_{s\bar{s}}(s) = \exp\{-i K(\xi) + i c_1)$$

where c_1 is to be determined.

The case of two solitons of equal charge (ss or $\bar{s}\bar{s}$) can be handled in the same way. The corresponding two soliton solution

$$u(x, t \mid p_1, q_1, p_2, q_2, +, +) =$$

$$= 4 \operatorname{arctg}\left(\operatorname{cth} \frac{\psi_1 - \psi_2}{2} \; \frac{\operatorname{ch} \frac{d_1 - d_2}{2}}{\operatorname{sh}\left(\frac{d_1 + d_2}{2} - \ln \operatorname{th} \frac{\psi_1 - \psi_2}{2} \right)} \right)$$

also decays into the sum of two one-soliton solutions when $|t| \to \infty$. There is however a difference: maxima in the asymptotics for $t \to -\infty$ approach one another as time increases, stop at some distance from one another, depending on their initial velocities, and then go backwards. This behavior corresponds to repulsion at small distances. The net result of this classical scattering is the same as in the first example but with the out coordinates interchanged. The S-matrix takes the form

$$\langle p_1', p_2' | S_{ss} | p_1, p_2 \rangle = \delta(p_1 - p_2')\,\delta(p_2' - p_1)\, S_{ss}(s)$$

$$S_{ss}(s) = \exp\{-i K(\xi) + i c_2\}$$

The constant c_2 is not necessarily equal to c_1.

We now discuss the crossing properties of the S-matrix. The expression for $S(s)$ that we found has a spurious cut in the gap $0 \le s \le 4M^2$ due to an accumulation of poles, corresponding to the bound states in $s\bar{s}$ scattering. In fact, their number $N = \frac{8\pi}{\gamma}$ becomes infinite in our approximation. This obstacle prevents us from directly comparing the analytic continuation of the forward $s\bar{s}$ scattering amplitude with the backward ss scattering amplitude. S. Coleman (28) has proposed the following indirect way to do the comparison. Hermitian unitarity requires that $S(s)$ below the cut is the complex conjugate of $S(s)$ on the physical side of the cut. The crossing condition can be written as follows

$$S_{s\bar{s}}(4M^2 - s + i0) = S^*_{ss}(s + i0)$$

where the LHS is to be defined by the analytic continuation of $S_{s\bar{s}}(s)$ via the upper half plane. One can check that this formula is correct if

$$c_1 - c_2 = \frac{8\pi^2}{\gamma} .$$

R. Jackiw and C. Woo[15] and V. E. Korepin[29], have calculated the phase shift, by computing the classical action of the two-soliton solution. They find that

$$c_1 = \frac{8\pi^2}{\gamma}, \quad c_2 = 0$$

thus confirming that crossing is indeed true.

Let us once again examine the S-matrix written in the form

$$S(s) = \exp\left\{ \frac{N}{\pi} \int_0^{\pi} d\theta \ln \frac{\xi + e^{i\theta}}{\xi e^{i\theta} + 1} \right\}$$

where we have substituted for the coupling constant γ, its expression in terms of the number of bound states, N. This formula suggests that the higher order corrections to the S-matrix "quantitizes" the exponential in the following way

$$\frac{N}{\pi} \int_0^{\pi} d\theta \ln \frac{\xi + e^{i\theta}}{\xi e^{i\theta} + 1} \rightarrow \sum_n^N \ln \frac{\xi + e^{i\theta_n}}{\xi e^{i\theta_n} + 1}$$

$$\theta_n = \frac{\pi}{N} n \quad ; \qquad n = 1, \dots, N$$

The S-matrix will then have the form

$$S(s) = \prod_n \frac{\xi + e^{i\theta_n}}{\xi e^{i\theta_n} + 1}$$

so that the spurious cut now disappears and instead S(s) has poles in

$$\xi_n = e^{i(\pi - \theta_n)} \quad ; \qquad s_n = 4M^2 \sin^2 \frac{n\pi}{2N}$$

thus confirming the conjectured formula for bound state masses. A perturbation procedure analogous to that of § 2 can be used to prove this conjecture. One must introduce two constraints in the functional integral before doing a stationary phase approximation, to take into account the independent conservation of energy and momentum.

This section ends with several comments. First, we shall present one more argument in support of the mass formula for the bound states. One can compute the action of the classical double soliton solution which gives a classical-tree approximation for the exponential of the Green function for the double soliton. The result is

$$F_{-1}(p, \alpha, T) = \exp\left\{ -i\, 2M \sin\alpha \sqrt{1-v^2}\, T \right\} U(t_1, t_2 \mid p, \alpha)$$

where $p = \dfrac{2M \sin\alpha\, v}{\sqrt{1-v^2}}$ and U is quasiperiodic in the c-m system (p = 0)

$$U(t_1 + \tau, t_2 \mid 0, \alpha) = e^{i\varepsilon}\, U(t_1, t_2 \mid 0, \alpha)$$

$$\tau = \frac{2\pi}{m \cos\alpha} \quad , \qquad \varepsilon = \frac{32\pi\alpha}{\gamma}$$

If we do not want to modify the coefficient in front of T in the above exponential we must set

$$\frac{32\pi\alpha}{\gamma} = 2\pi n \quad , \qquad \alpha_n = \frac{\gamma}{16} n \, , \qquad n < \frac{8\pi}{\gamma}$$

which gives the mass formula written above. The next comment concerns scattering. One can find the classical-tree approximation for the S-matrix of several solitons, double-soliton-double-soliton scattering, or for double soliton-soliton scattering etc. in a way analogous to that used for soliton-soliton scattering. We present several formulae found by V. E. Korepin (29) without derivation. The double soliton-double soliton S-matrix is given by

$$S_{\alpha_i \alpha_k}(s) = \exp\left\{ iK\left(\xi e^{i(\alpha_i-\alpha_k)}\right) + iK\left(\xi e^{i(\alpha_k-\alpha_i)}\right) + iK\left(-\xi e^{-i(\alpha_i+\alpha_k)}\right) + iK\left(-\xi e^{i(\alpha_i+\alpha_k)}\right) \right\},$$

where

$$\xi = \frac{s - M_i^2 - M_k^2}{2 M_i M_k} + \sqrt{\frac{s - M_i^2 - M_k^2}{2 M_i M_k} - 1}$$

and $K(\xi)$ was defined above. Double soliton-soliton scattering is given by

$$S_{d s_k}(s) = \exp\left\{ iK\left(\xi e^{i(\frac{\pi}{2}-\alpha_k)}\right) + iK\left(\xi e^{i(\alpha_k-\frac{\pi}{2})}\right)\right\},$$

$$\xi = \frac{s - M^2 - M_k^2}{2 M M_k} + \sqrt{\frac{s - M^2 - M_k^2}{2 M M_k} - 1} .$$

The fundamental particle-soliton S-matrix is given by

$$S(s) = \frac{\sqrt{m^2 - \chi} + m}{\sqrt{m^2 - \chi} - m}, \quad \chi = \left(\frac{s - m^2 - M^2}{2M}\right)^2$$

which completes the list of S-matrices calculated so far.

4. CONCLUSIONS AND SPECULATIONS

We can extract from the preceding arguments the following attractive features of field theoretic models with soliton sectors.

1. One can calculate perturbatively. The physical quantities--i.e. masses, phase-shifts--are nonanalytic in the coupling constant, but all singular terms are given by classical computations. Quantum corrections are analytic in the coupling constant.

2. Soliton-like excitations interact strongly when the fundamental field interacts weakly. Indeed, the masses and phase shifts computed above have the coupling constant appearing in the denominator.

3. Soliton-like excitations have a nontrivial charge, Q of topological origin.

It is very important to demonstrate that the same general features can be realized in a 3-dimensional model. The existence of solitons--namely finite evergy, static or periodic in time, localized solutions of classical field equations--would be a first indication that the quantum field under consideration has nontrivial particle-like excitations. Several examples of the existence of such solutions are known (31), (32), (33), (8), (9), (20) and their number is increasing. We shall now discuss the properties of field theoretic models that lead to the existence of topological charge.

One can generalize the one-dimensional case in two ways. The first will be a natural generalization of the φ^4 model and the second will be that for the sine-Gordon system.

The one-dimensional current

$$J_\mu = \varepsilon_{\mu\nu} \partial_\nu u$$

generating the charge Q has the same form in both cases, but admits different interpretations and higher dimensional generalizations. In the φ^4 model with the self interaction

$$\mathcal{L}_{int} = \left(\varphi^2(x) - \lambda^2\right)^2$$

the field φ can have the asymptotic values $\pm\lambda$ when $|x| \to \infty$. The interpretation is that φ defines a map of the boundary of the space, consisting of just two points $X = \pm\infty$ into the two-point set (a, -a).

$$\varphi : \begin{pmatrix} \infty \\ -\infty \end{pmatrix} \to \begin{pmatrix} a \\ -a \end{pmatrix}$$

The charge Q is trivial if $\varphi(\infty) = \varphi(-\infty)$, i.e. if the map is into one point, and nontrivial if $\varphi(\infty) \neq \varphi(-\infty)$, i.e. if the map is onto both of them. This is easily generalized to higher dimensions. Consider the Higgs field ψ^a, $a = 1, \ldots, n$, where n is the dimension of space, with the self interaction

$$\mathcal{L}_{int} = \left(\psi_a \psi_a - \lambda^2\right)^2$$

Different vacua are parameterized by points on the n-1-dimensional sphere S^{n-1}. The field ψ^a, defines the map of the boundary S^{n-1} of the space R^n into the manifold of vacua

$$\psi^a : S^{n-1} \to S^{n-1} .$$

The corresponding charge Q is a homotopy class of such a map and can take arbitrary integer values in contrast with the case $n = 1$. All the necessary elementary topological notions can be found in (24). The charge Q can be calculated by integrating the current

$$J_\mu = \varepsilon_{\mu\nu\rho\sigma} \varepsilon^{abc} \partial_\nu \psi^a \partial_\rho \psi^b \partial_\sigma \psi^c ,$$

written here for $n = 3$. It is easy to see that J_0 is a divergence so that the charge is nontrivial only if the Higgs fields have nontrivial asymptotics. The examples

$$\psi \to f(\rho)\, e^{i\varphi} \quad , \quad \psi^2 \to f(r) \frac{x^a}{r}$$

of a complex field for $n = 2$ and an isovector real field for $n = 3$ were used in (31) and (9), (33), respectively.

The sine-Gordon charge has a different interpretation. The field $\chi(x)$ can be considered as a map from the space R^1 into a nonlinear internal space--the circle S^1

$$\chi : R^1 \to S^1$$

with the prescribed vacuum value

$$\chi(-\infty) = \chi(\infty) = 1 .$$

Such a map is characterized by integers called winding numbers which is our charge Q. The generalization to higher dimensions requires the use of the nonlinear chiral field χ, for example

$$\chi : R^n \to S^n,\ \chi\big|_{|x| \to \infty} \to \chi_0 .$$

The corresponding topological charge is a homotopy class of this map and can be computed by integrating the current

$$J_\mu = \varepsilon_{\mu\nu\rho\sigma} \varepsilon^{abc} \partial_\nu \pi^a \partial_\rho \pi^b \partial_\sigma \pi^c \frac{1}{(1+\pi^2)^3}$$

(see i.e. (30)) where we use the usual parametrization of χ-field in the form

$$\chi = (\pi^a, \sigma),\ \pi^a \pi^a + \sigma^2 = 1 .$$

Notice that in contrast with a Higgs type model this J_0 is not a divergence and the charge density for this type of charge can be localized arbitrarily. It is this feature that makes the second type of topological charge a more attractive one. The use of this type of charge was advocated by T. H. R. Skyrme and D. Finkelstein who seem to have been the first to discuss the possible role of topology in field theory. I must admit however that the cost of using chiral fields is the nonrenormalizability of the corresponding field theoretic models in the usual sense of the term.

The example $\chi : R^3 \to S^3$ is neither typical nor minimal for a chiral field with topological charge in three dimensions. The minimal one is

$$n : R^3 \to S^2$$

that is, n is a scalar isovector field $n = (n^1, n^2, n^3)$, satisfying a constraint

$$n_1^2 + n_2^2 + n_3^2 = 1 \quad .$$

The topological charge for this field is called a Hopf invariant in mathematics and it can not be written in terms of a simple local current. Nevertheless it has a kind of locality.

I have used the n-field in a particular model of electromagnetic and weak interactions of leptons,[34] where it played the role of a neutral direction in the three-dimensional internal space of electric charge. It was

shown in (32) that this model admits localized static solutions in the case of a two-dimensional space. The field n, realizes the map

$$n : R^2 \to S^2$$

and has a nontrivial charge Q associated with the current analogous to that written above for $n = 3$. This solution can be interpreted as an infinite vortex in the three-dimensional case. One can speculate that the corresponding closed vorticies could be candidates for localized solutions in the case $n = 3$. Usually such vorticies are unstable as they have a tendency to shrink in order to decrease their energy. To make them stable one can introduce angular momentum, attach fermion fields to the string etc. The last speculation in this paper is to show that these modifications might not be necessary for a vortex made out of the n-field. The Hopf invariant of a closed vortex will be nontrivial if we twist the vortex around the core by a full 2π rotation before closing it. Such a twisted vortex can be stable and static because resistance against the twist for a small radius can overwhelm the tendency the vortex has to shrink. So far, I have not been able to find a suitable Ansatz or provide a variational estimate, to prove that this is indeed the case. Work in this direction is in progress.

REFERENCES

1. L. D. Faddeev, L. A. Takhtajan, Uspekhy Mat. Sci. 29, 249 (1974).
2. L. D. Faddeev, L. A. Takhtajan, Theor. Math. Phys. 21, 160 (1974), (see also preprint JINR, E27998 (1974).
3. V. E. Zakharov, L. A. Takhtajan, L. D. Faddeev, Doklady Akad. Sci. USSR, 219, 1334 (1974).
4. I. Ya. Arefieva, V. E. Korepin, Pizma JETP 20, 680 (1974)
5. L. D. Faddeev, Pizma JETP, 21, 141 (1975).
6. L. D. Faddeev, P. P. Kulish, V. E. Korepin, Pizma JETP, 21, 302 (1975).
7. V. E. Korepin, L. D. Faddeev, Theor. Math. Phys. (to be published).
8. R. Dashen, B. Hasslacher, A. Neveu, Phys. Rev. D10, 4114; 4130; 4138 (1974).
9. A. Polyakov, Pizma JETP, 20, 6 (1974); JETP to be published.
10. J. Goldstone, R. Jackiw, Phys. Rev. D11, 1486 (1975).
11. R. Dashen, B. Hasslacher, A. Neveu, IAS preprint, COO 2220-37, (1975).
12. J. Gervais, R. Sakita, CCNY preprint, Dec. 1974.
13. S. Coleman, Phys. Rev. D11, to be published (1975).
14. S. Mandelstam, Berkeley University preprint, Feb. 1975.
15. R. Jackiw, G. Woo, MIT preprint N 469, Apr. 1975.
16. C. S. Callan, D. J. Gross, Princeton University preprint.
17. N. H. Christ, T. D. Lee, Columbia University preprint, CO-2271-55.
18. R. Dashen, B. Hasslacher, A. Neveu, IAS preprint, COO 2220-46 (1975).

19. S. J. Chang, S. D. Ellis, B. W. Lee, preprint FERMILAB-Pub-75/22 THY, Feb. 1975.
20. W. A. Bardeen, M. S. Chanowitz, S. D. Drell, M. Weinstein, T.-M. Yan, Phys. Rev. D11, 1094 (1975).
21. A. Scott, F. Chu, D. McLaughlin, Proc. IEEE 61, 1443 (1973).
22. V. E. Zakharov, S. V. Manakov, Theor. Math. Phys. 19, 332 (1974).
23. "Mathematical Physics in One Dimension" ed. E. Lieb, D. Mattis, Academic Press, New York, London, 1966.
24. D. Finkelstein, J. Math. Phys. 7, 1216 (1966).
25. L. D. Faddeev, Theor. Math. Phys. 1, 3 (1969).
26. L. D. Faddeev, V. S. Buslaev, Doklady Acad. Sci. USSR, 132, 13 (1960).
27. S. Coleman, R. Jackiw, H. S. Politzer, Phys. Rev. D10, 2491 (1974).
28. S. Coleman, private communication.
29. V. E. Korepin, private communication.
30. T. H. R. Skyrme, Proc. Roy. Soc. (London) A247, 260 (1958), A260, 127 (1961), A262, 237 (1961).
31. H. Nielsen, P. Olesen, Nucl. Phys. B61, 45 (1973).
32. L. D. Faddeev, Max Planck Inst. preprint MPI-PAE/Pth 16, June, 1974.
33. G. 't Hooft, Nucl. Phys. B79, 276 (1974).
34. L. D. Faddeev, Doklady Acad. Sci. USSR, 210, 807 (1973).

FOOTNOTE

I wish to thank the Institute for Advanced Study for their kind hospitality while I was visiting Princeton.

For copies of this preprint please write to the Institute for Advanced Study, Princeton, New Jersey 08540.